Appendices to "Effect of Schmidt number on the velocity–scalar cospectrum in isotropic turbulence with a mean scalar gradient"

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Journal of Fluid Mechanics, vol. 532 (2005), pp. 111–140

Appendix C

Here we will give the derivations of equations (B 26) and (B 28). Our starting point is the exact equation (B 9) for the evolution of $\tilde{W}_i(\mathbf{k},t,t)$. Decomposing the velocity and scalar fields, and making use of assumptions (i) and (ii), the two triple correlations in equation (B 9) may be written as

$$-i\,k_j\left(\frac{2\pi}{L}\right)^6\sum_{\mathbf{p}}\sum_{\mathbf{q}}{}_{(\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}\Big[\,\overline{\tilde{u}_j^{(1)}(-\mathbf{p},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{\theta}^{(0)}(-\mathbf{q},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{u}_i^{(0)}(-\mathbf{k},t\|\mathbf{k},\mathbf{p},\mathbf{q})}$$
$$+\overline{\tilde{u}_j^{(0)}(-\mathbf{p},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{\theta}^{(1)}(-\mathbf{q},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{u}_i^{(0)}(-\mathbf{k},t\|\mathbf{k},\mathbf{p},\mathbf{q})}$$
$$+\overline{\tilde{u}_j^{(0)}(-\mathbf{p},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{\theta}^{(0)}(-\mathbf{q},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{u}_i^{(1)}(-\mathbf{k},t\|\mathbf{k},\mathbf{p},\mathbf{q})}\,\Big] \tag{C 1}$$

$$+\frac{i}{2}\left(\frac{2\pi}{L}\right)^6\tilde{P}_{ijm}(\mathbf{k})\sum_{\mathbf{p}}\sum_{\mathbf{q}}{}_{(\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}\Big[\,2\,\overline{\tilde{u}_j^{(1)}(\mathbf{p},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{u}_m^{(0)}(\mathbf{q},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{\theta}^{(0)}(\mathbf{k},t\|\mathbf{k},\mathbf{p},\mathbf{q})}$$
$$+\overline{\tilde{u}_j^{(0)}(\mathbf{p},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{u}_m^{(0)}(\mathbf{q},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{\theta}^{(1)}(\mathbf{k},t\|\mathbf{k},\mathbf{p},\mathbf{q})}\,\Big]. \tag{C 2}$$

Note that we choose $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$ as the triad of non-interacting wavenumbers for each term in the double summation. Consider the first term of (C 1). We can substitute for the velocity deviation field using the following expression derived in KG,

$$\tilde{u}_i^{(1)}(\mathbf{k},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0) = i\,\frac{(2\pi)^9}{L^6}\,\tilde{P}_{abc}(\mathbf{k})\int_{t_0}^{t}dt'\,\tilde{G}_{ia}^{(E0)}(\mathbf{k},t|-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)$$
$$\times\big[\,-\delta^3_{\mathbf{k}-\mathbf{k}_0}\,\tilde{u}_b^{(0)}(-\mathbf{p}_0,t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{u}_c^{(0)}(-\mathbf{q}_0,t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)$$
$$-\,\delta^3_{\mathbf{k}+\mathbf{k}_0}\,\tilde{u}_b^{(0)}(\mathbf{p}_0,t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{u}_c^{(0)}(\mathbf{q}_0,t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)$$
$$+\,(\mathbf{k}_0\to\mathbf{p}_0\to\mathbf{q}_0\to\mathbf{k}_0)\,\big]\,, \tag{C 3}$$

so that,

$$
\begin{aligned}
&[\text{first term of (C 1)}] = \\
&k_j \, \frac{(2\pi)^{15}}{L^{12}} \sum_{\substack{\mathbf{p} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \sum_{\mathbf{q}} \tilde{P}_{abc}(\mathbf{p}) \int_{t_0}^{t} dt' \, \overline{\tilde{G}^{(E0)}_{ja}(-\mathbf{p}, t|\mathbf{p}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q})} \\
&\overline{\times \, \tilde{u}^{(0)}_b(\mathbf{k}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{u}^{(0)}_c(\mathbf{q}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{\theta}^{(0)}(-\mathbf{q}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{u}^{(0)}_i(-\mathbf{k}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q})}.
\end{aligned}
\tag{C 4}
$$

This is rewritten using assumption (ii) as,

$$
\begin{aligned}
&[\text{first term of (C 1)}] = \\
&= k_j \, \frac{(2\pi)^{15}}{L^{12}} \sum_{\substack{\mathbf{p} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \sum_{\mathbf{q}} \tilde{P}_{abc}(\mathbf{p}) \int_{t_0}^{t} dt' \, \overline{\tilde{G}^{(E0)}_{ja}(-\mathbf{p}, t|\mathbf{p}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q})} \\
&\times \, \overline{\tilde{u}^{(0)}_b(\mathbf{k}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{u}^{(0)}_i(-\mathbf{k}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q})} \; \overline{\tilde{u}^{(0)}_c(\mathbf{q}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{\theta}^{(0)}(-\mathbf{q}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q})}.
\end{aligned}
\tag{C 5}
$$

Next we make use of (E 3) from Appendix E, and results derived in KG,

$$
\tilde{Q}_{ij}(\mathbf{k}, t, t') = \left(\frac{2\pi}{L}\right)^3 \overline{\tilde{u}^{(0)}_i(\mathbf{k}, t\|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0) \, \tilde{u}^{(0)}_j(-\mathbf{k}, t'\|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)}, \tag{C 6}
$$

$$
\overline{\tilde{G}^{(E0)}_{ij}(\mathbf{k}, t|\mathbf{k}', t'\|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = \overline{\tilde{G}^{(L0)}_{ij}(t|\mathbf{k}, \mathbf{k}', t'\|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)}, \tag{C 7}
$$

$$
\frac{(2\pi)^6}{L^3} \, \overline{\tilde{G}^{(L0)}_{im}(t|\mathbf{k}, -\mathbf{k}, t'\|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} \, \tilde{P}_{mj}(\mathbf{k}) = \tilde{G}_{ij}(\mathbf{k}, t, t'), \tag{C 8}
$$

to find

$$
\begin{aligned}
[\text{first term of (C 1)}] = k_j \left(\frac{2\pi}{L}\right)^3 \sum_{\substack{\mathbf{p} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \sum_{\mathbf{q}} \int_{t_0}^{t} dt' \, \tilde{Q}_{ib}(-\mathbf{k}, t, t') \, \tilde{W}_c(-\mathbf{q}, t, t') \\
\times \left[p_c \, \tilde{G}_{jb}(-\mathbf{p}, t, t') + p_b \, \tilde{G}_{jc}(-\mathbf{p}, t, t') \right].
\end{aligned}
\tag{C 9}
$$

Turning now to the second term of (C 1), we perform a similar procedure, and begin by substituting for $\tilde{\theta}^{(1)}(-\mathbf{q}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q})$ according to equation (B 25). We use assumption (i) to change $\tilde{u}_j$ to $\tilde{u}^{(0)}_j$, and also make use of assumption (ii) to find

$$
\begin{aligned}
&[\text{second term of (C 1)}] \\
&= k_j \, \frac{(2\pi)^{15}}{L^{12}} \sum_{\substack{\mathbf{p} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \sum_{\mathbf{q}} q_l \int_{t_0}^{t} dt' \, \overline{\tilde{G}^{(0)}(-\mathbf{q}, t|\mathbf{q}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q})} \\
&\times \Big[\overline{\tilde{u}^{(0)}_j(-\mathbf{p}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{u}^{(0)}_l(\mathbf{p}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q})} \; \overline{\tilde{\theta}^{(0)}(\mathbf{k}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{u}^{(0)}_i(-\mathbf{k}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q})} \\
&\quad + \overline{\tilde{u}^{(0)}_l(\mathbf{k}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{u}^{(0)}_i(-\mathbf{k}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q})} \; \overline{\tilde{\theta}^{(0)}(\mathbf{p}, t'\|\mathbf{k}, \mathbf{p}, \mathbf{q}) \, \tilde{u}^{(0)}_j(-\mathbf{p}, t\|\mathbf{k}, \mathbf{p}, \mathbf{q})} \Big].
\end{aligned}
\tag{C 10}
$$

We now use equation (E 4) from Appendix E and a result derived in Goto & Kida (1999),

$$
\overline{\tilde{G}^{(0)}(\mathbf{k}, t|-\mathbf{k}, t'\|\mathbf{k}_0, \mathbf{p}_0, \mathbf{q}_0)} = \overline{\tilde{G}^{(L)}(t|\mathbf{k}, -\mathbf{k}, t')}, \tag{C 11}
$$

to find that

$$[\text{second term of (C 1)}] = k_j \, \frac{(2\pi)^9}{L^6} \sum_{\substack{\mathbf{p} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \sum_{\mathbf{q}} q_l \int_{t_0}^{t} dt' \, \overline{\tilde{G}^{(L)}(t|-\mathbf{q},\mathbf{q},t')} \times \Big[\tilde{Q}_{jl}(-\mathbf{p},t,t')\,\tilde{X}_i(-\mathbf{k},t,t') + \tilde{Q}_{il}(-\mathbf{k},t,t')\,\tilde{X}_j(-\mathbf{p},t,t')\Big]. \tag{C 12}$$

It remains to find a more useful expression for $\overline{\tilde{G}^{(L)}(t|\mathbf{k},-\mathbf{k},t')}$. From equation (B 15) we can write

$$\begin{aligned}
\frac{\partial}{\partial t}\overline{\tilde{G}^{(L)}(t|\mathbf{k},-\mathbf{k},t')} &= -\kappa\,\frac{(2\pi)^6}{L^3}\sum_{\mathbf{p}} p^2\,\overline{\tilde{G}(\mathbf{p},t|-\mathbf{k},t')\,\tilde{\psi}(-\mathbf{p},t|\mathbf{k},t')} \\
&= -\kappa\,\frac{(2\pi)^6}{L^3}\sum_{\mathbf{p}} p^2\,\overline{\tilde{G}^{(0)}(\mathbf{p},t|-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(-\mathbf{p},t|\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)} \\
&= -\kappa\,k^2\,\overline{\tilde{G}^{(0)}(\mathbf{k},t|-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)} \\
&= -\kappa\,k^2\,\overline{\tilde{G}^{(L)}(t|\mathbf{k},-\mathbf{k},t')},
\end{aligned} \tag{C 13}$$

where we have used assumptions (i) and (iii), and (E 2) and (C 11). This can be solved with initial condition (B 16) to give

$$\overline{\tilde{G}^{(L)}(t|\mathbf{k},-\mathbf{k},t')} = \frac{L^3}{(2\pi)^6}\,\exp[-\kappa\,k^2\,(t-t')]. \tag{C 14}$$

Therefore,

$$[\text{second term of (C 1)}] = k_j \left(\frac{2\pi}{L}\right)^3 \sum_{\substack{\mathbf{p} \\ (\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}} \sum_{\mathbf{q}} q_l \int_{t_0}^{t} dt' \, \exp[-\kappa\,q^2\,(t-t')] \times \Big[\tilde{Q}_{jl}(-\mathbf{p},t,t')\,\tilde{X}_i(-\mathbf{k},t,t') + \tilde{Q}_{il}(-\mathbf{k},t,t')\,\tilde{X}_j(-\mathbf{p},t,t')\Big]. \tag{C 15}$$

Performing a similar procedure on the remaining terms in (C 1) and (C 2) results in equation (B 26).

Turning now to the evolution of the two-time correlation $\tilde{W}_i(\mathbf{k},t,t')$, we can approximate the diffusive term in (B 10) as

$$\begin{aligned}
&-\kappa\,\frac{(2\pi)^9}{L^6}\sum_{\mathbf{p}} p^2\,\overline{\tilde{\theta}(\mathbf{p},t)\,\tilde{\psi}(-\mathbf{p},t|\mathbf{k},t')\,\tilde{u}_i(-\mathbf{k},t')} \\
&= -\kappa\,\frac{(2\pi)^9}{L^6}\sum_{\mathbf{p}} p^2\,\overline{\tilde{\theta}^{(0)}(\mathbf{p},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(-\mathbf{p},t|\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{u}_i^{(0)}(-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)} \\
&= -\kappa\left(\frac{2\pi}{L}\right)^3 k^2\,\overline{\tilde{\theta}^{(0)}(\mathbf{k},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{u}_i^{(0)}(-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)} \\
&= -\kappa\,k^2\,\tilde{W}_i(\mathbf{k},t,t'),
\end{aligned} \tag{C 16}$$

where we have used assumptions (i) and (iii), and equations (E 2) and (E 3). Substituting into equation (B 10) leads to (B 28).

Appendix D

Here we will outline the derivation of equation (B 29). We begin with the evolution equation (B 11) for $\tilde{Y}_i(\mathbf{k},t,t')$. Making the DIA decompositions for the velocity, position function, and scalar fields, the viscous term becomes

$$\begin{aligned}
&-\nu\,\frac{(2\pi)^9}{L^6}\sum_{\mathbf{p}} p^2\,\overline{\tilde{\theta}(-\mathbf{k},t')\,\tilde{u}_i(\mathbf{p},t)\,\tilde{\psi}(-\mathbf{p},t|\mathbf{k},t')} \\
&= -\nu\,\frac{(2\pi)^9}{L^6}\sum_{\mathbf{p}} p^2\,\overline{\tilde{\theta}^{(0)}(-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{u}_i^{(0)}(\mathbf{p},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(-\mathbf{p},t|\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)} \\
&= -\nu\left(\frac{2\pi}{L}\right)^3 k^2\,\overline{\tilde{\theta}^{(0)}(-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{u}_i^{(0)}(\mathbf{k},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)} \\
&= -\nu\,k^2\,\tilde{X}_i(\mathbf{k},t,t'),
\end{aligned} \tag{D 1}$$

where we have used assumptions (i) and (iii), and equations (E 2) and (E 4). Again using assumptions (i), (iii), and (E 2) we see that the quadruple correlation in (B 11) leads to three terms proportional to k_i (containing a deviation field $\tilde{u}_m^{(1)}(\mathbf{p},t)$, $\tilde{u}_n^{(1)}(\mathbf{q},t)$, and $\tilde{\theta}^{(1)}(-\mathbf{k},t')$ respectively), a term with no deviation field that is zero by assumption (ii), and the following term,

$$\begin{aligned}
&-i\,\frac{(2\pi)^{12}}{L^9}\sum_{\mathbf{p}}\sum_{\mathbf{q}}\sum_{\mathbf{r}}{}_{(\mathbf{p}+\mathbf{q}+\mathbf{r}=0)}\,\frac{r_i\,r_m\,r_n}{r^2} \\
&\times\overline{\tilde{\theta}^{(0)}(-\mathbf{k},t'\|\mathbf{r},\mathbf{p},\mathbf{q})\,\tilde{u}_m^{(0)}(\mathbf{p},t\|\mathbf{r},\mathbf{p},\mathbf{q})\,\tilde{u}_n^{(0)}(\mathbf{q},t\|\mathbf{r},\mathbf{p},\mathbf{q})\,\tilde{\psi}^{(1)}(\mathbf{r},t|\mathbf{k},t'\|\mathbf{r},\mathbf{p},\mathbf{q})}.
\end{aligned} \tag{D 2}$$

We now substitute for $\tilde{\psi}^{(1)}(r,t|\mathbf{k},t'\|\mathbf{r},\mathbf{p},\mathbf{q})$ with the following equation derived in KG,

$$\begin{aligned}
\tilde{\psi}^{(1)}(\mathbf{k},t|\mathbf{k}',t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0) = -i\,k_j\,\frac{(2\pi)^9}{L^6}\int_{t'}^{t} dt''\,\tilde{\psi}^{(0)}(\mathbf{k},t|-\mathbf{k},t''\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0) \\
\times\,[\;\delta^3_{\mathbf{k}-\mathbf{k}_0}\,\tilde{u}_j^{(0)}(-\mathbf{p}_0,t''\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(-\mathbf{q}_0,t''|\mathbf{k}',t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0) \\
+\,\delta^3_{\mathbf{k}-\mathbf{k}_0}\,\tilde{u}_j^{(0)}(-\mathbf{q}_0,t''\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(-\mathbf{p}_0,t''|\mathbf{k}',t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0) \\
+\,\delta^3_{\mathbf{k}+\mathbf{k}_0}\,\tilde{u}_j^{(0)}(\mathbf{p}_0,t''\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(\mathbf{q}_0,t''|\mathbf{k}',t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0) \\
+\,\delta^3_{\mathbf{k}+\mathbf{k}_0}\,\tilde{u}_j^{(0)}(\mathbf{q}_0,t''\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(\mathbf{p}_0,t''|\mathbf{k}',t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0) \\
+\,(\mathbf{k}_0\to\mathbf{p}_0\to\mathbf{q}_0\to\mathbf{k}_0)\;]\,.
\end{aligned} \tag{D 3}$$

Using assumption (iii) and equation (E 2), and after some algebra, expression (D 2) becomes

$$\begin{aligned}
&-2\,\frac{(2\pi)^9}{L^9}\sum_{\mathbf{p}}\sum_{\mathbf{q}}{}_{(\mathbf{k}+\mathbf{p}+\mathbf{q}=0)}\,\frac{q_i\,q_m\,q_n\,q_j}{q^2}\int_{t'}^{t} dt'' \\
&\times\overline{\tilde{\theta}^{(0)}(-\mathbf{k},t'\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{u}_n^{(0)}(\mathbf{k},t\|\mathbf{k},\mathbf{p},\mathbf{q})}\;\overline{\tilde{u}_m^{(0)}(\mathbf{p},t\|\mathbf{k},\mathbf{p},\mathbf{q})\,\tilde{u}_j^{(0)}(-\mathbf{p},t''\|\mathbf{k},\mathbf{p},\mathbf{q})}.
\end{aligned} \tag{D 4}$$

It is convenient to take the incompressible projection of equation (B 11), so that terms proportional to k_i drop out, and use of (C 6) and (E 4) results in equation (B 29).

Appendix E

Here we will relate $\tilde{W}_i(\mathbf{k},t,t')$ and $\tilde{X}_i(\mathbf{k},t,t')$ to $\tilde{\theta}^{(0)}(\mathbf{k},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)$ and $\tilde{u}_i^{(0)}(\mathbf{k},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)$. Starting with the definition (4.12) of $\tilde{W}_i(\mathbf{k},t,t')$, and using (B 8) and assumption (i), we can write

$$\begin{aligned}\tilde{W}_i(\mathbf{k},t,t') &= \frac{(2\pi)^9}{L^6}\sum_{\mathbf{k}'}\overline{\tilde{\theta}(\mathbf{k}',t)\,\tilde{\psi}(-\mathbf{k}',t|\mathbf{k},t')\,\tilde{u}_i(-\mathbf{k},t')}\\ &= \frac{(2\pi)^9}{L^6}\sum_{\mathbf{k}'}\overline{\tilde{\theta}^{(0)}(\mathbf{k}',t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(-\mathbf{k}',t|\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)}\\ &\quad\overline{\times\tilde{u}_i^{(0)}(-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)}.\end{aligned} \tag{E 1}$$

Then, using assumption (iii) and a result from KG,

$$\overline{\tilde{\psi}^{(0)}(\mathbf{k},t|\mathbf{k}',t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)} = \frac{L^3}{(2\pi)^6}\,\delta^3_{\mathbf{k}+\mathbf{k}'}, \tag{E 2}$$

gives

$$\tilde{W}_i(\mathbf{k},t,t') = \left(\frac{2\pi}{L}\right)^3\overline{\tilde{\theta}^{(0)}(\mathbf{k},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{u}_i^{(0)}(-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)}. \tag{E 3}$$

Similarly for $\tilde{X}_i(\mathbf{k},t,t')$ defined by (4.15),

$$\begin{aligned}\tilde{X}_i(\mathbf{k},t,t') &= \frac{(2\pi)^9}{L^6}\,\tilde{P}_{ia}(\mathbf{k})\sum_{\mathbf{k}'}\overline{\tilde{u}_a(\mathbf{k}',t)\,\tilde{\psi}(-\mathbf{k}',t|\mathbf{k},t')\,\tilde{\theta}(-\mathbf{k},t')}\\ &= \frac{(2\pi)^9}{L^6}\,\tilde{P}_{ia}(\mathbf{k})\sum_{\mathbf{k}'}\overline{\tilde{u}_a^{(0)}(\mathbf{k}',t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\psi}^{(0)}(-\mathbf{k}',t|\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)}\\ &\quad\overline{\times\tilde{\theta}^{(0)}(-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)}\\ &= \left(\frac{2\pi}{L}\right)^3\overline{\tilde{u}_i^{(0)}(\mathbf{k},t\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)\,\tilde{\theta}^{(0)}(-\mathbf{k},t'\|\mathbf{k}_0,\mathbf{p}_0,\mathbf{q}_0)}.\end{aligned} \tag{E 4}$$

Note that we consistently have $t \geq t'$, and so equations (E 3) and (E 4) do not mean that $\tilde{W}_i(\mathbf{k},t,t')$ and $\tilde{X}_i(\mathbf{k},t,t')$ are equivalent in this approximation.